Chapter Six:
Relations, Functions, and Quantifiers

6.1. Introduction: More Logical Form

1. Relations and Relation Letters. Once more we expand the formal language. Our motivation for doing so is again to insure that the formal test of validity rightly evaluates intuitively valid arguments. The following argument, for example, strikes us as valid.

> 1. For any objects x, y: x is the same age as y if and only if y is the same age as x.
> 2. Ace is the same age as Rex
>
> ---
>
> ∴ Rex is the same age as Ace.

We could manage a translation of the second premise and conclusion using the following translation table.

A: Ace	**G**: ____ is the same age as Rex
B: Rex	**H**: ____ is the same age as Ace

> 2. GA
>
> ---
>
> ∴ HB

But consider the odd-looking ‘predicates’ in the first premise: is x said to be “*the same age as y*,” and y said to be “*the same age as x*”? Supposing that’s right, citing those odd-looking predicates in the translation table yields the following formal translation.

1. For any objects x, y: x is the same age as y if and only if y is the same age as x.
2. Ace is the same age as Rex

$\therefore$ Rex is the same age as Ace.

A: Ace	**G**: ____ is the same age as Rex
B: Rex	**H**: ____ is the same age as Ace
	I: ____ is the same age as *y*
	J: ____ is the same age as *x*

1. $\forall$x $\forall$y(Ix $\leftrightarrow$ Jy)
2. GA

$\therefore$ HB

The argument form is susceptible to simple validity counterexamples, such as the following model.

U: {2, 3}

A: 2	
B: 3	
G: {2}	**I**: {2, 3}
H: {2}	**J**: {2, 3}

The first premise is true: since every object here is both I and J, in this model it's true of *any* objects that one is I and the other is J. With the left and right parts of "(Ix $\leftrightarrow$ Jy)" true regardless of which objects we consider, the biconditional holds for all objects in the model.

The second premise is true as well. But the conclusion is false – making the formal argument **invalid**.

We recognize a familiar problem here: the formal translation is overlooking common features in the English original. For all four 'predicates' contain "*is the same age as*". But the predicate letters "G," "H," "I," and "J" reflect those overlaps not at all.

Properly modifying the language to recognize the common component "*is the same age as*" brings in items more complex than the predicates of the previous chapter. For a predicate such as "*is made of steel*" or "*is a cat*" has one "place" (or 'blank') to be filled in (by a name), in order to yield a complete sentence, capable of truth or falsehood. Now "*is the same age as*" certainly has such a blank that can be filled in by a proper name – yielding, e.g., the "*Ace is the same age as*" and "*Rex is the same age as*" appearing in the above translation table. But even with a name added we aren't left with a complete sentence: "*Ace is the same age as*" is a mere fragment of a sentence, and is (fittingly) neither truth nor false.

What's needed for a complete sentence is, of course, mention of who or what Ace is the same age as – another name. And sure enough, adding a name yields a complete sentence, making a claim that is either true or false – say, the second premise, "*Ace is the same age as* Rex".

The string of words "*is the same age as*" is, we now see, different from a predicate such as "*is made of steel*" in the **number of "places"** (blanks) it needs filled to provide a complete sentence: while the predicates of Chapter Five were **one** name short of a complete sentence, "*is the same age as*" is **two** names away from forming a complete sentence. Reserving the term "predicate" for our **one-place** sentence-makers, we will call a two-place sentence-maker (such as "*is the same age as*") a "**relation phrase**" of English.[1]

Recognizing that "*is the same age as*" has two blanks to be filled, we make sense as well of those earlier odd pseudo-predicates "*is the same age as* x" and "*is the same age as* y". Since variables typically appear in the same contexts as do name letters, we are here again filling in the second blank of "*is the same age as*" (now with a variable) while again leaving its first blank empty.

To recognize the common feature in all the sentences of the above argument – the relation phrase "*is the same age as*" – our formal language needs a formal counterpart to such an English string of words. So besides our earlier one-place predicate letters, we now introduce two-place **relation letters**. The same letter that count as predicate letters will be pressed into service as

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[1] Using "phrase" very loosely here, to mean a string of words. A string such as "*is the same age as*" won't count as a **grammatical** phrase of English, in the sense of being a 'natural part'(a "constituent") of an English sentence.

relation letters as well – though making clear the two places to be filled with the numerical superscript “**2**”. We will likewise mark a letter as a (mere) predicate letter with a superscript “**1**”. Our translation tables will then look like the following example.

> $\mathbf{G^1}$: ____ is Greek
> $\mathbf{H^2}$: ____ is the same age as ____

And just as the subject followed a predicate letter, both blank-filling formal terms will follow a relation letter.

> **A**: Ace $\mathbf{G^2}$: ____ is the same age as ____
> **B**: Rex
>
> **Ace is the same age as Rex**: $\mathbf{G^2AB}$

Armed with relation letters, we can at last formally represent the earlier argument in a way that recognizes the common ground among its three sentences.

> 1. For any objects x, y: x is the same age as y if and only if y is the same age as x.
> 2. Ace is the same age as Rex
>
> ---
>
> ∴ Rex is the same age as Ace.
>
> **A**: Ace $\mathbf{G^2}$: ____ is the same age as ____
> **B**: Rex
>
> 1. 1. $\forall x\, \forall y(G^2xy \leftrightarrow G^2yx)$
> 2. G^2AB
>
> ---
>
> ∴ G^2BA

2. Relation Letters Extended. Once we’ve recognized that the predicate letters of old can be extended to more than one place (‘blank’), we recognize as well that there’s no need to stop at two-place relations. For English speaks naturally of **three-place relations** as well.

A: Pittsburgh **G^3**: ____ is between ____ and _____
B: Indianapolis
C: Philadelphia

Pittsburgh is between Indianapolis and Philadelphia.

G^3ABC

(The "and" might tempt us to treat this sentence as a *conjunction* of two smaller claims. But we should resist that temptation: the claim that "*Pittsburgh is between Indianapolis and Philadelphia*" is **not** a conjunction of the two smaller **nonsensical** sentences "*Pittsburgh is between Indianapolis*" and "*Pittsburgh is between Philadelphia*".)

Further examples are provided in sentences such as "

And once we see that relation letters can be opened to any (integral) number of places, a further possibility suggests itself. Understanding the number of places in a relation to be the number of names which need to be added to yield a complete sentence, we can understand as well the concept of a **zero-place relation letter**. This would be a (capital) letter requiring no names added in order to qualify as a complete formal sentence.

That describes a **sentence letter**: a letter constituting a complete formal sentence on its own, with no need for added name letters.

Recognizing the sentence letters of old as zero-place relation letters, we allow formal talk of predicates and relations to range over the capital letters G through Z – in each case specifying the number of places with the appropriate superscript numeral.

> **Relation letters (including sentence letters and predicate letters):** capital letters G through Z (with or without numerical subscript, and with a numerical superscript).

3. Function Letters. Augmented with relation letters, our formal language can adequately translate the following sentences.

A: Socrates **G^2**: ____ is the teacher of ____
B: Plato
C: Aristotle

Socrates is the teacher of Plato: **G^2AB**

Plato is the teacher of Aristotle: **G^2BC**

But it's at least cumbersome to translate the following sentence.

Socrates is the teacher of the teacher of Aristotle.
(*More naturally*: **Socrates is the teacher of Aristotle's teacher**)

Perhaps the best we do at the moment is the following existential sentence.[2]

$$\exists x(G^2Ax \wedge G^2xC)$$

We can manage a far simpler translation, however, through a further extension of the Chapter Five language. For note that allowing for any number of places in relation letters allowed us to see sentence letters in a new light, as just the limit case (zero-place version) of relation letters.

But the same holds true for names (and name letters): they could be viewed as just the simplest, zero-place version of a broader family of referring phrases. Whereas a proper name has zero 'blanks' to be filled before it refers to some individual, its one-place counterpart is a phrase with one blank to be filled (by, e.g., a proper name) before it successfully refers.

An example comes in the English "*the teacher of*". As it stands it fails to refer to an individual. But if a proper name such as "*Plato*" is added, we get a genuine referring phrase – "*the teacher of Plato*" which picks out an individual just as the proper name "*Socrates*" does.

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[2] In our discussion of *definite descriptions* in X.xx, we sharpen this translation to more correctly reflect "the" in "the teacher of".

To accommodate such incomplete referring phrases into the formal language, we extend its (zero-place) name letters to one-place **function letters**. (Once again, we note the number of places involved by a superscript numeral "**1**".)

> **A**: Plato $\quad$ $\mathbf{G^1}$: is Greek ____
> **B**: Aristotle
> $\mathbf{C^1}$: the teacher of ____

"*The teacher of Plato*" is then translated as "$\mathbf{C^1A}$," and "the teacher of Aristotle" as "$\mathbf{C^1B}$".

And note that we can treat function letters recursively: for since $\mathbf{C^1B}$" is a referring phrase on a par with a name letter, it too can fill the 'blank' in "$\mathbf{C^1}$__". The more complex phrase "*the teacher of the teacher of Aristotle*" then becomes "$\mathbf{C^1C^1B}$". The sentence "The teacher of the teacher of Aristotle is Greek" ("The teacher of Aristotle's teacher is Greek") is translated as follows.

> **The teacher of the teacher of Aristotle is Greek**: $\mathbf{G^1\ C^1C^1B}$

Chapter Six Construction Rules

Atomic Terms:

T1. Name letters are atomic terms

T2. Variables are atomic terms

Terms:

1. Atomic terms are terms
2. A function letter with *n* many places, followed by *n* many terms, is a term.

Atomic Formulas:

A1. A relation letter with *n* many places, followed by *n* many terms, is an atomic formula.

Formulas:

1. Atomic formulas are formulas.
2. If ● is a formula, then ~● is a formula.
3. If ● and ▲ are formulas, then (● ∧ ▲) is a formula.
4. If ● and ▲ are formulas, then (● ∨ ▲) is a formula.
5. If ● and ▲ are formulas, then (● →▲) is a formula.
6. If ● and ▲ are formulas, then (● ↔▲) is a formula.
7. If ★ is a variable and ● is a formula, then

∃★ ●

and

∀★ ●

are both formulas.